Pre-class Warm-up!!!

Let f : R^3 -> R be a function.
Select the best answer to complete the sentence.

The gradient of f is

a. a function R^3 -> R

b. a function R -> R^3

c, a function R^3 -> R^3 ✓

d, not defined.

e, none of the above.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

Second question:
What color was Wednesday's
Pre-class Warm-up!!! ?

4.3 Vector fields

We learn:

- What is a vector field
- Examples:
  flow of a fluid
  force fields
  gradient vector fields
- Flow lines

Types of question:

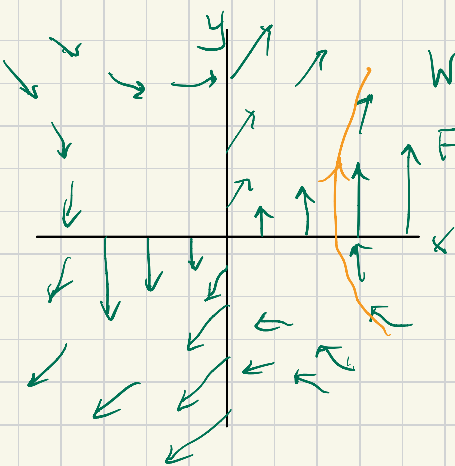
- sketch and recognize vector fields
- Verify that a given path is a flow line for some vector field
- Find a function with a specified vector field as gradient (qn 21, but not done in the text of the book)

Things we don't do (right now):

- Escape velocity
- Newton's gravitational law
- Coulomb's law
- Show that a vector field is not a gradient vector field (example 7)

Definition: a vector field on $\mathbb{R}^n$ is a function $F: \mathbb{R}^n \to \mathbb{R}^n$.

Sketch the vector field $F(x,y) = (y, x+y)$



Plot points.

When $y=0$

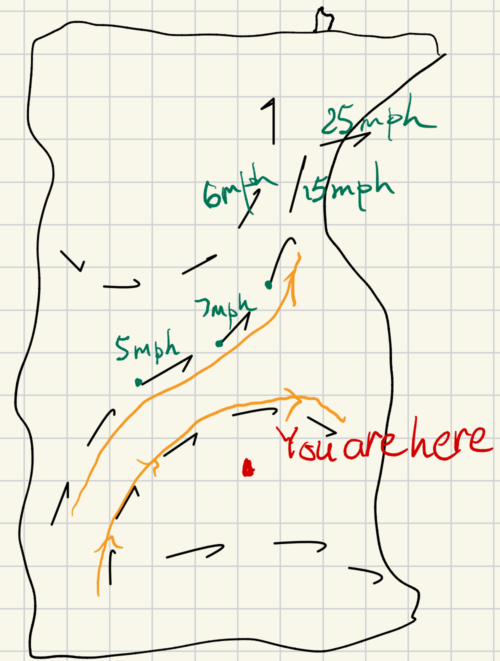
$F(x,0) = (0,x)$

When $x=0$

$F(0,y) = (y,y)$

Definition: a flow line for a vector field F is a path $c: \mathbb{R} \to \mathbb{R}^n$ so that

$c'(t) = F(c(t))$

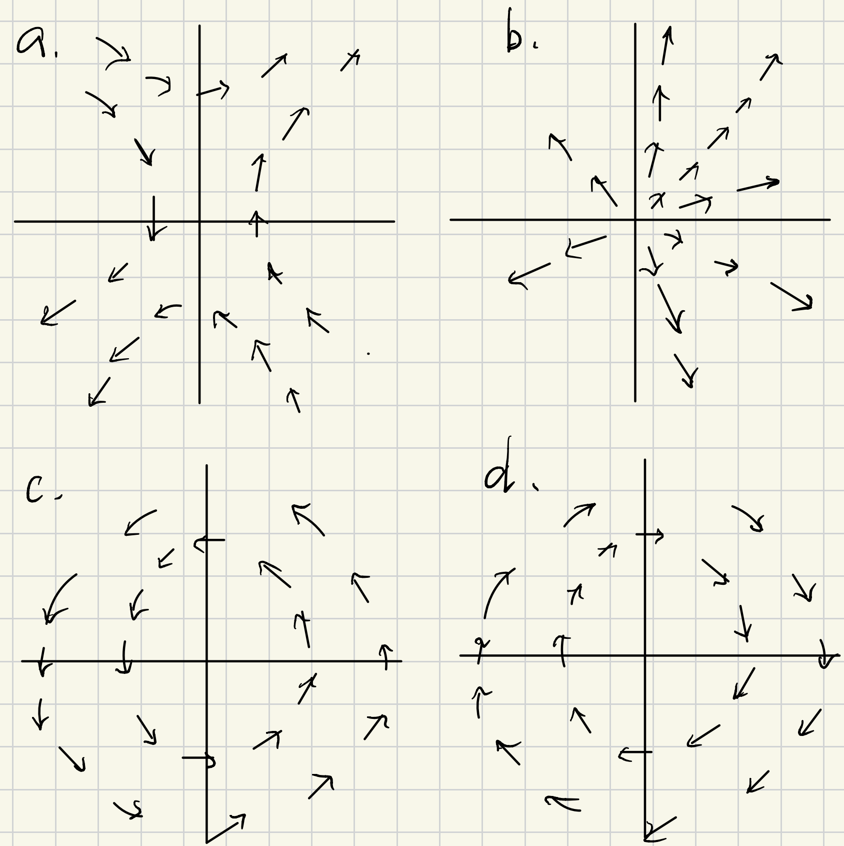


Wind velocities in MN are described by a vector field.

Match the fields to the pictures

1. $F(x,y) = (y, -x)$ d
2. $F(x,y) = (-y,x)$ c
3. $F(x,y) = (x,y)$ b
4. $F(x,y) = (y,x)$ a

In 1. when $y=0$, $F(x,0)=(0,-x)$ which points down on the positive x-axis

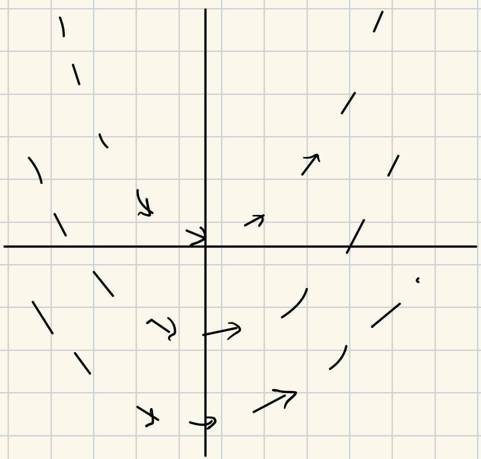


Like questions 15 - 20:
Show that c(t) = (t, t^2 / 2) is a flow line for the vector field F(x,y) = (1,x).

Solution. We check $c'(t) = F(c(t))$ always.

$c'(t) = (1, t)$. $F(c(t)) = (1, t)$

These are equal so c is a flow line. $\square$



Like question 21. Find a function f so that F(x,y,z) = (y^2, 2xy, 1) is the gradient of f (or show that such f does not exist).

Solution. We want $f: \mathbb{R}^3 \to \mathbb{R}$

so $\frac{\partial f}{\partial x} = y^2$, $\frac{\partial f}{\partial y} = 2xy$, $\frac{\partial f}{\partial z} = 1$

Then equation 1 says $f = xy^2 + a(y,z)$

Equation 2 says $f = xy^2 + b(x,z)$

Equation 3 says $f = z + c(x,y)$

a, b, c are unknown functions

$f = xy^2 + z$ is a required function

4.4 Curl and divergence

We learn:

- The definitions of
  div F when F : R^n -> R^n
  curl F when F : R^3 -> R^3
- Notation $\operatorname{div} F = \nabla \cdot F$, $\operatorname{curl} F = \nabla \times F$
  $\operatorname{grad} f = \nabla f$
- Physical interpretations
- curl (grad f) = 0 and div (curl F) = 0
- the Laplacian. $\nabla \cdot (\nabla f) = \nabla^2 f$.

What you don't need to memorize:

- the other formulas on page 255.

Types of questions:

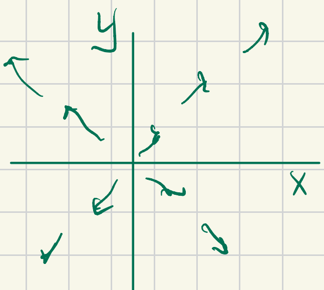
- calculate div and curl.
- Which composites make sense?
- Verify e.g. curl (grad f) = 0
- Scalar curl.

Definition Let F = (F_1, … , F_n). The divergence of F is

$$\operatorname{div} F = \nabla\cdot F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}$$

$$\nabla\cdot F : \mathbb{R}^n \longrightarrow \mathbb{R}$$

Examples: F(x,y,z) = (x,y,0) ; G(x,y,z) = (-y,x,0)

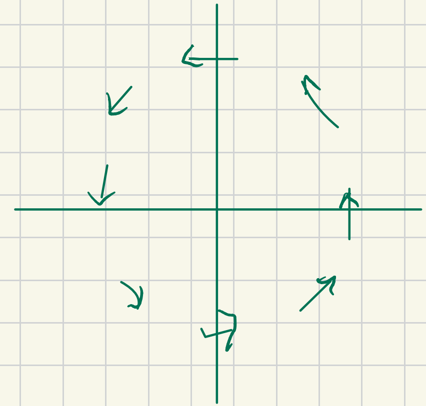


$$\nabla\cdot F = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial 0}{\partial z}$$

$$= 1 + 1 + 0 = 2$$

for all (x,y,z)

$\nabla\cdot F$ measures the amount of stuff being produced by F



$$\nabla\cdot G = \frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial 0}{\partial z} = 0$$

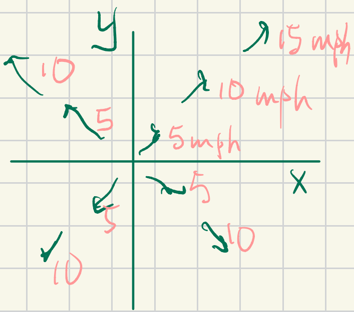
Definition Let F = (F_1, F_2 , F_3). The curl of F is

$$\text{curl } F = \nabla \times F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= \det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

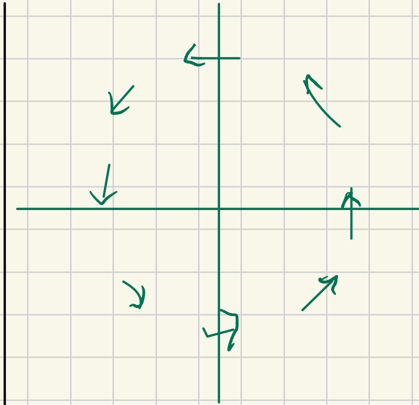
Examples: F(x,y,z) = (x,y,0) ; G(x,y,z) = (-y,x,0)



$$\nabla \times F$$

$$= \left(\frac{\partial 0}{\partial y} - \frac{\partial y}{\partial z}, \quad , \quad \right)$$

$$= (0, 0, 0)$$



$$\nabla \times G = (0, 0, 2)$$

$$= \left(\frac{\partial 0}{\partial y} - \frac{\partial x}{\partial z}, \frac{\partial (-y)}{\partial z} - \frac{\partial 0}{\partial x}, \frac{\partial x}{\partial x} - \frac{\partial (-y)}{\partial y} \right)$$

curl measures counterclockwise rotation about a vector, and points in the direction of that vector